Teoría de Expansiones Asintóticas Acopladas

Imaginemos que tenemos que resolver el siguiente problema adimensionalizado :

$$\epsilon u'' + u u' + u^2 = 0 \qquad 0 \leq x \leq 1 \ ; \ \epsilon \ll 1$$

$$x = 0 : u = 0$$
$$x = 1 : u = 1$$

Solución de orden 0 o de Euler $(\epsilon = 0)$:

$$u u' + u^2 = 0 \quad \text{ORDEN } 1 \longrightarrow \underline{\text{SÓLO 1 C.C. EN } x}$$

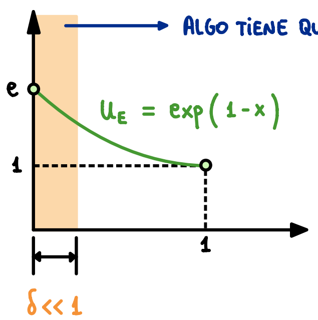
$$\begin{cases} u_E = 0 \ \times \\ u_E' + u_E = 0 \ \checkmark \longrightarrow u_E = \boxed{C_1} \exp(-x) \end{cases}$$

Si imponemos $x = 0 : u = 0 \longrightarrow x = 0 : u_E = 0 \longrightarrow C_1 = 0 \ \times$

Si imponemos $x = 1 : u = 1 \longrightarrow x = 1 : u_E = 1 \longrightarrow C_1 = e \ \checkmark \longrightarrow \boxed{u_E = \exp(1-x)}$

DESTRUYE LA HOMOGENEIDAD

$$\left.\begin{array}{l} u_E(x=0) = e \\ u_E(x=1) = 1 \end{array}\right\} \longrightarrow \frac{u_E(x=0) - u_E(x=1)}{u_E(x=0)} \sim O(1) \longrightarrow \text{PROBLEMA DE PERTURBACIONES SINGULARES (PPS)}$$



ALGO TIENE QUE PASAR AQUÍ PARA CUMPLIR LA C.C. EN $x = 0$ $\longrightarrow$ ACTIVAR $\epsilon u''$ $\longrightarrow$

$\longrightarrow$ NUEVA ESCALA δ $\left(\frac{\delta}{\ell} \ll 1 \right)$ CERCA DE $x = 0$.

Definimos $\xi = \dfrac{x}{\delta(x)}$ de manera que cuando las variaciones de x sean de orden δ, las variaciones de ξ sean de orden unidad $\Delta x \sim \delta \longrightarrow \Delta \xi \sim 1$ (ZOOM).

Por tanto :

$$\left.\begin{array}{l} u' = \dfrac{du}{d\xi} \underbrace{\dfrac{d\xi}{dx}}_{1/\delta} = \dfrac{1}{\delta} \dfrac{du}{d\xi} \\[2ex] u'' = \dfrac{d}{dx} \left(\dfrac{1}{\delta} \dfrac{du}{d\xi} \right) = \dfrac{1}{\delta^2} \dfrac{d^2 u}{d\xi^2} \end{array}\right\} \longrightarrow \frac{\epsilon}{\delta^2} \frac{d^2 u}{d\xi^2} + \frac{u}{\delta} \frac{du}{d\xi} + u^2 = 0$$

Multiplicando por δ :

$$\underbrace{\frac{\epsilon}{\delta}\overbrace{\frac{d^2u}{d\xi^2}}^{\sim 1}}_{\boxed{\sim \frac{\epsilon}{\delta}}} + \underbrace{u\overbrace{\frac{du}{d\xi}}^{\sim 1}}_{\sim 1} + \underbrace{\delta u^2}_{\sim \delta} = 0 \longrightarrow \text{3 CASOS}$$

- $\frac{\epsilon}{\delta} \ll 1 \longrightarrow$ Volvemos a Euler ✗
- $\frac{\epsilon}{\delta} \sim 1 \longrightarrow \underbrace{\frac{\epsilon}{\delta}\frac{d^2u}{d\xi^2}}_{\text{ACTIVADO}} + \underbrace{u\frac{du}{d\xi}}_{\text{PERMITE EMPALMAR CON EULER}} = 0$ ✓ LÍMITE DISTINGUIDO DE δ
- $\frac{\epsilon}{\delta} \gg 1 \longrightarrow$ término descompensado, no podemos conectar con Euler ✗

Por tanto :

$$\delta(\epsilon) \sim \epsilon \xrightarrow{\text{Para simplificar}} \boxed{\delta(\epsilon) = \epsilon}$$

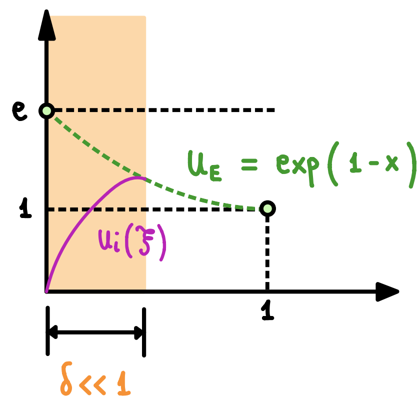
$$\underbrace{\frac{d^2u}{d\xi^2}}_{\sim 1} + \underbrace{u\frac{du}{d\xi}}_{\sim 1} + \underbrace{\cancel{\epsilon u^2}}_{\sim \epsilon \ll 1} = 0 \xrightarrow{u|_{x\sim\delta\sim\epsilon} = u_i}$$

$\longrightarrow$

$$\boxed{\begin{array}{l} \dfrac{d^2u_i}{d\xi^2} + u_i\dfrac{du_i}{d\xi} = 0 \\ \xi = 0 : \; u_i = 0 \\ \xi \gg 1 : \; u_i \rightarrow u_E|_{x\to 0} = e \end{array}}$$

EMPALME CON EULER CERCA DE $X = 0$:

$$u_i\Big|_{\substack{\xi\to\infty \\ (\xi \gg 1)}} = u_E\Big|_{\substack{x\to 0 \\ (x \ll 1)}}$$



Truco : $\frac{du_i}{d\xi} = q \longrightarrow \frac{d^2u_i}{d\xi^2} = \frac{dq}{d\xi} = \frac{dq}{du_i}\underbrace{\frac{du_i}{d\xi}}_{\xi} = \xi\frac{dq}{du_i} \longrightarrow \xi\frac{dq}{du_i} + u_i q = 0 \longrightarrow$

$$\longrightarrow q\left(\frac{dq}{du_i} + u_i\right) = 0 \quad \begin{cases} q = 0 \xrightarrow{\xi = 0 \,:\, u = 0} u_i = 0 \; ✗ \\ \dfrac{dq}{du_i} = -u_i \longrightarrow q = a - \dfrac{u_i^2}{2} = a\left[1 - \left(\dfrac{u_i}{\sqrt{2a}}\right)^2\right] = \dfrac{du_i}{d\xi} \longrightarrow \end{cases}$$

$$\longrightarrow \int \frac{d\left(\frac{u_i}{\sqrt{2a}}\right)}{1 - \Big(\underbrace{\frac{u_i}{\sqrt{2a}}}_{\varphi}\Big)^2} = \int \sqrt{\frac{a}{2}}\, d\xi \xrightarrow{\frac{1}{1-\blacksquare^2} = \frac{1}{2}\left(\frac{1}{1-\blacksquare} + \frac{1}{1+\blacksquare}\right)} \underbrace{\frac{1}{2}\int\frac{\varphi}{1-\varphi}\,d\varphi}_{-\ln(1-\varphi)} + \underbrace{\frac{1}{2}\int\frac{\varphi}{1+\varphi}\,d\varphi}_{\ln(1+\varphi)} \longrightarrow$$

$$\longrightarrow \ln\left(\frac{1+\varphi}{1-\varphi}\right) = \sqrt{2a}\,\xi + \ln b \longrightarrow \frac{1+\varphi}{1-\varphi} = b\exp\left(\sqrt{2a}\,\xi\right)$$

Para hallar a y b recurrimos a las condiciones de contorno :

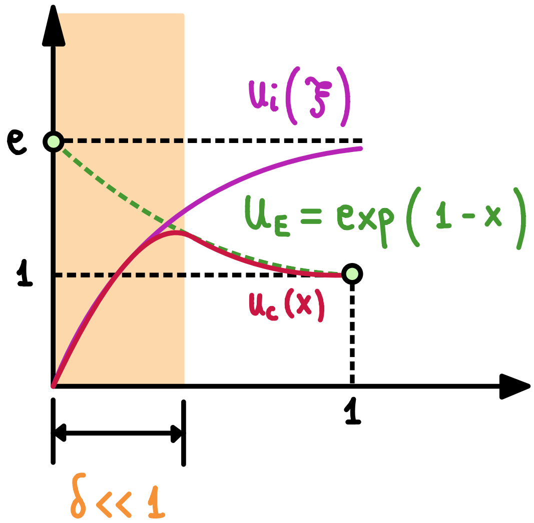
$$\xi = 0 : \; u_i = 0 \longrightarrow \varphi = 0 \longrightarrow \boxed{b = 1}$$

$$\left.\begin{array}{l} \xi \gg 1 : \; u_i \rightarrow u_E|_{x\to 0} = e \; \begin{cases} \varphi \rightarrow \dfrac{e}{\sqrt{2a}} \\ b\exp\left(\sqrt{2a}\,\xi\right) \rightarrow \cancel{\infty} \quad \varphi = 1 \text{ LO BLOQUEA} \end{cases} \end{array}\right\} \longrightarrow \boxed{a = \frac{e^2}{2}}$$

Entonces :

$$\frac{1 + \frac{u_i}{e}}{1 - \frac{u_i}{e}} = \exp(e\xi) \longrightarrow \boxed{\frac{e + u_i}{e - u_i} = \exp(e\xi)}$$

Solución compuesta : $u_c(x) = u_E(x) + u_i(\xi) - \underbrace{u_E\big|_{x\to 0}}_{u_e = e} \longrightarrow \boxed{u_c(x) = u_E(x) + u_i\left(\frac{x}{\epsilon}\right) - e}$



Comprobación de validez :

$$x \sim 1 : u_c(x) = u_E(x) + \cancel{\underbrace{u_i(\xi >> 1)}_{u_e}} - \cancel{u_e} = u_E(x) \checkmark$$

$$x >> 1 : u_c(x) = \cancel{\underbrace{u_E(x << 1)}_{u_e}} + u_i\left(\frac{x}{\epsilon}\right) - \cancel{u_e} = u_i\left(\frac{x}{\epsilon}\right) \checkmark$$